

Inverse scattering on the line for the matrix Sturm-Liouville equation

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Abstract. The inverse scattering problem is studied for the matrix Sturm-Liouville equation on the line. Necessary and sufficient conditions for the scattering data are obtained.

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1. Introduction

In this paper, we consider the matrix Sturm-Liouville (also called Schrödinger) equation on the real line:

$$-Y'' + Q(x)Y = \rho^2 Y, \quad -\infty < x < \infty. \quad (1.1)$$

Here $Y = [y_k(x)]_{k=1}^m$ is a vector function, ρ is the spectral parameter, and $Q(x) = [Q_{jk}(x)]_{j,k=1}^n$ is the self-adjoint matrix potential ($Q(x) = Q^*(x)$), satisfying the condition

$$\int_{-\infty}^{\infty} (1 + |x|) |Q_{jk}(x)| dx < \infty, \quad j, k = \overline{1, m}. \quad (1.2)$$

The inverse scattering problem is studied, which consists in recovering of the potential Q from the given scattering data.

There is an extensive literature on the inverse spectral and scattering problems (see monographs [1, 2, 3, 4, 5] and references therein). Such problems arise in quantum mechanics, geophysics, electronics, chemistry and other branches of science and engineering. One of the most important applications is the inverse scattering method for integration of nonlinear evolution equations, such as Korteweg-de Vries (KdV) equation, nonlinear Schrödinger equation, sine-Gordon equation, Toda lattice, etc. (see [6, 7, 8, 9]).

A complete analysis of the inverse scattering problem for the *scalar* Sturm-Liouville equation on the real line (equation (1.1) for $m = 1$) was carried out by L.D. Faddeev [10] and V.A. Marchenko [1]. They have obtained the characterization of the scattering data, in other words, necessary and sufficient conditions for the solvability of the inverse scattering problem.

Inverse scattering problems for *matrix* Sturm-Liouville operators appeared to be more difficult, than scalar ones, because of the more complicated structure of the discrete scattering data. For the matrix case, Z.S. Agranovich and V.A. Marchenko [11] solved the inverse scattering problem of the *half-line*, using the transformation operator method [1, 2]. Later on, several authors studied the inverse scattering problem for the matrix Sturm-Liouville operator on the *line* [12, 13, 14, 15, 16, 17, 18]. In particular, M. Wadati and T. Kamijo [12] reduced the inverse problem for the self-adjoint potential to the Gelfand-Levitan-Marchenko equation (linear Fredholm integral equation, connecting the scattering data with the kernel of the transformation operator). E. Olmedilla [16] generalized their results to the non-self-adjoint case. F. Calogero and A. Degasperis [13] study applications of the inverse scattering problem to the integration of matrix nonlinear evolution equations such as matrix KdV and Boomeron equations. However, as far as we know, a rigorous mathematical analysis of the solvability of the Gelfand-Levitan-Marchenko equation and characterization of the scattering data were not done before for the matrix case. The goal of this paper is to cover this gap, and provide necessary and sufficient conditions for the solvability of the inverse scattering problem for equation (1.1).

The paper is organized as follows. In Section 2, we introduce the left and the right scattering data for equation (1.1), study their properties, formulate the inverse scattering problem and give other preliminaries. In Section 3, the Gelfand-Levitan-Marchenko equation is derived. Although many of the results of Sections 2 and 3 appeared before in [11, 12, 20] and other literature, we provide their proofs for the convenience of the reader. In Section 4, we give sufficient conditions for the unique solvability of the Gelfand-Levitan-Marchenko equation. However, these conditions are not sufficient for the solvability of the inverse scattering problem, because the solution of the main equation, constructed, for example, by the right scattering data, is guaranteed to fulfill (1.2) only on the right half-line. Therefore we have to study the connection between the left and the right scattering data. In Section 5, we formulate and prove our main result, Theorem 5.3 on the necessary and sufficient conditions on the scattering data of the matrix Sturm-Liouville operator. We show how the left scattering data can be constructed by the right ones. Then the resulting potential satisfies (1.2) on the full line, and we prove that its scattering data coincide with the given ones. We also investigate the case of reflectionless potentials with only discrete scattering data, and obtain the characterization for them as a corollary of the main theorem. Finally, we discuss a connection of our necessary and sufficient conditions with the Riemann problem, and apply them to the matrix KdV equation.

Let us introduce the notation. We consider the Hilbert space of complex column m -vectors \mathbb{C}^m with the following scalar product and the norm

$$(Y, Z) = \sum_{j=1}^m \overline{y_j} z_j, \quad \|Y\| = \left(\sum_{j=1}^m |y_j|^2 \right)^{1/2}, \quad Y = [y_j]_{j=\overline{1,m}}, \quad Z = [z_j]_{j=\overline{1,m}},$$

the space of row vectors $\mathbb{C}^{m,T}$, and the space of complex $m \times m$ matrices $\mathbb{C}^{m \times m}$ with the corresponding induced norm

$$\|A\| = \max_{Y \in \mathbb{C}^m, \|Y\|=1} \|AY\|, \quad A = [a_{jk}]_{j,k=\overline{1,m}}.$$

The symbols I_m and 0_m are used for the unit $m \times m$ matrix and the zero $m \times m$ matrix, respectively. The symbol “ $*$ ” denotes the conjugate transpose.

We use the notation $\mathcal{A}(\mathcal{I}; \mathbb{C}^m)$, $\mathcal{A}(\mathcal{I}; \mathbb{C}^{m,T})$ and $\mathcal{A}(\mathcal{I}; \mathbb{C}^{m \times m})$ for classes of column vectors, row vectors and matrices, respectively, with entries belonging to the class $\mathcal{A}(\mathcal{I})$ of scalar functions. The symbol \mathcal{I} stands for an interval. For example, the potential Q belongs to the class $L((-\infty, \infty); \mathbb{C}^{m \times m})$.

2. Properties of the scattering data

2.1. Jost solutions. Let us introduce matrix Jost solutions of equation (1.1) with the prescribed behavior at $\pm\infty$. Equation (1.1) has unique matrix solutions $F_{\pm}(x, \rho)$ for $\text{Im } \rho \geq 0$, such that $\lim_{x \rightarrow \pm\infty} \exp(\mp i\rho x) F_{\pm}(x, \rho) = I_m$. The Jost solutions $F_{\pm}(x, \rho)$ satisfy the integral equations

$$\begin{aligned} F_+(x, \rho) &= \exp(i\rho x) I_m + \int_x^{\infty} \frac{\sin \rho(t-x)}{\rho} Q(t) F_+(t, \rho) dt, \\ F_-(x, \rho) &= \exp(-i\rho x) I_m + \int_{-\infty}^x \frac{\sin \rho(x-t)}{\rho} Q(t) F_-(t, \rho) dt, \end{aligned} \tag{2.1}$$

and have the following **properties**.

(i_1) For each fixed x , the matrix functions $F_{\pm}^{(\nu)}(x, \rho)$, $\nu = 0, 1$, are analytic for $\text{Im } \rho > 0$ and continuous for $\text{Im } \rho \geq 0$.

(i_2) For $\nu = 0, 1$

$$\begin{cases} F_+^{(\nu)}(x, \rho) = (i\rho)^{\nu} \exp(i\rho x) (I_m + o(1)), & x \rightarrow +\infty, \\ F_-^{(\nu)}(x, \rho) = (-i\rho)^{\nu} \exp(-i\rho x) (I_m + o(1)), & x \rightarrow -\infty, \end{cases} \tag{2.2}$$

uniformly in $\{\rho: \text{Im } \rho \geq 0\}$.

(i₃) For each fixed ρ , $\text{Im } \rho \geq 0$, and real a ,

$$F_+(x, \rho) \in L_2((a, \infty); \mathbb{C}^{m \times m}), \quad F_-(x, \rho) \in L_2((-\infty, a); \mathbb{C}^{m \times m}).$$

Moreover, every vector solution of (1.1) from $L_2((a, \infty); \mathbb{C}^m)$ (or $L_2((-\infty, a); \mathbb{C}^m)$) can be represented in the form $F_+(x, \rho)V(\rho)$ (or $F_-(x, \rho)V(\rho)$, respectively), where $V(\rho) \in \mathbb{C}^m$.

(i₄) For $|\rho| \rightarrow \infty$, $\text{Im } \rho \geq 0$, $\nu = 0, 1$,

$$\begin{cases} F_+^{(\nu)}(x, \rho) = (i\rho)^\nu \exp(i\rho x) \left(I_m + \frac{\omega_+(x)}{i\rho} + o(\rho^{-1}) \right), & \omega_+(x) = -\frac{1}{2} \int_x^\infty Q(t) dt, \\ F_-^{(\nu)}(x, \rho) = (-i\rho)^\nu \exp(-i\rho x) \left(I_m + \frac{\omega_-(x)}{i\rho} + o(\rho^{-1}) \right), & \omega_-(x) = -\frac{1}{2} \int_{-\infty}^x Q(t) dt, \end{cases} \quad (2.3)$$

uniformly for $x \geq a$ and $x \leq a$, respectively.

(i₅) For real $\rho \neq 0$ the columns of the matrix functions $F_+(x, \rho)$ and $F_+(x, -\rho)$ (similarly, $F_-(x, \rho)$ and $F_-(x, -\rho)$) form a fundamental system of solutions for equation (1.1).

(i₆) There are transformation operators for the Jost solutions:

$$\begin{cases} F_+(x, \rho) = \exp(i\rho x) I_m + \int_x^\infty K_+(x, t) \exp(i\rho t) dt, \\ F_-(x, \rho) = \exp(-i\rho x) I_m + \int_{-\infty}^x K_-(x, t) \exp(-i\rho t) dt, \end{cases} \quad (2.4)$$

where the kernels $K_\pm(x, t)$ are continuous functions, having first derivatives with respect to x and t , the matrix functions

$$\frac{\partial}{\partial x} K_\pm(x, t) \pm \frac{1}{4} Q \left(\frac{x+t}{2} \right), \quad \frac{\partial}{\partial t} K_\pm(x, t) \pm \frac{1}{4} Q \left(\frac{x+t}{2} \right)$$

are absolutely continuous with respect to x and t , and

$$Q(x) = \mp 2 \frac{d}{dx} K_\pm(x, x). \quad (2.5)$$

Note that, in fact, the Jost solutions are constructed for half-lines $(a, +\infty)$ and $(-\infty, a)$. So one can find the proofs of the properties (i₁)-(i₆), for example, in [11].

2.2. Matrix Wronskian. Together with equation (1.1), consider the following equation

$$-Z'' + ZQ(x) = \rho^2 Z, \quad -\infty < x < \infty, \quad (2.6)$$

where $Z = Z(x)$ is a row vector. Define the matrix Wronskian $\langle Z, Y \rangle := Z'Y - ZY'$. If $Y(x, \lambda)$ and $Z(x, \lambda)$ satisfy equations (1.1) and (2.6), respectively, then

$$\frac{d}{dx} \langle Z(x, \lambda), Y(x, \lambda) \rangle = 0, \quad (2.7)$$

so the expression $\langle Z(x, \lambda), Y(x, \lambda) \rangle$ does not depend on x .

One can introduce the Jost solutions $\bar{F}_\pm(x, \rho)$ for equation (2.6) with properties, similar to (i₁)-(i₆). Since the potential matrix $Q(x)$ is Hermitian, one can easily check that

$$\bar{F}_\pm(x, \rho) = F_\pm^*(x, -\bar{\rho}). \quad (2.8)$$

By virtue of (2.7) and (2.2)

$$\langle \bar{F}_\pm(x, \rho), F_\pm(x, \rho) \rangle = \lim_{x \rightarrow \pm\infty} (\bar{F}_\pm'(x, \rho) F_\pm(x, \rho) - \bar{F}_\pm(x, \rho) F_\pm'(x, \rho)) = 0_m, \quad (2.9)$$

$$\langle \bar{F}_\pm(x, \rho), F_\pm(x, -\rho) \rangle = \lim_{x \rightarrow \pm\infty} (\bar{F}'_\pm(x, \rho) F_\pm(x, -\rho) - \bar{F}_\pm(x, \rho) F'_\pm(x, -\rho)) = \pm 2i\rho I_m, \quad (2.10)$$

for real $\rho \neq 0$.

2.3. Scattering matrix. Since for real $\rho \neq 0$ the columns of the matrices $F_+(x, \rho)$ and $F_+(x, -\rho)$ (similarly, $F_-(x, \rho)$ and $F_-(x, -\rho)$) form a fundamental system of solutions for equation (1.1), the following relations hold

$$F_+(x, \rho) = F_-(x, -\rho)A(\rho) + F_-(x, \rho)B(\rho), \quad (2.11)$$

$$F_-(x, \rho) = F_+(x, \rho)C(\rho) + F_+(x, -\rho)D(\rho), \quad (2.12)$$

with the $m \times m$ matrix coefficients $A(\rho)$, $B(\rho)$, $C(\rho)$, $D(\rho)$. Let us study the properties of these coefficients.

Lemma 2.1. *The following relations hold for real $\rho \neq 0$:*

$$B^*(-\rho)A(\rho) = A^*(-\rho)B(\rho), \quad A^*(\rho)A(\rho) = I_m + B^*(\rho)B(\rho), \quad (2.13)$$

$$A(\rho) = D^*(-\rho), \quad B(\rho) = -C^*(\rho), \quad (2.14)$$

$$A(\rho) = -\frac{1}{2i\rho} \langle \bar{F}_-(x, \rho), F_+(x, \rho) \rangle, \quad (2.15)$$

$$B(\rho) = \frac{1}{2i\rho} \langle \bar{F}_-(x, -\rho), F_+(x, \rho) \rangle. \quad (2.16)$$

Proof. Using (2.11) and (2.8), we derive

$$\bar{F}_+(x, \rho) = A^*(-\rho)\bar{F}_-(x, -\rho) + B^*(-\rho)\bar{F}_-(x, \rho). \quad (2.17)$$

Consequently,

$$\begin{aligned} \langle \bar{F}_+(x, \rho), F_+(x, \rho) \rangle &= \langle A^*(-\rho)\bar{F}_-(x, -\rho) + B^*(-\rho)\bar{F}_-(x, \rho), F_-(x, -\rho)A(\rho) + F_-(x, \rho)B(\rho) \rangle \\ &= -2i\rho B^*(-\rho)A(\rho) + 2i\rho A^*(-\rho)B(\rho) = 0_m. \end{aligned}$$

Here we have applied (2.9) and (2.10). Similarly we obtain

$$\begin{aligned} \langle \bar{F}_+(x, \rho), F_+(x, -\rho) \rangle &= \langle A^*(-\rho)\bar{F}_-(x, -\rho) + B^*(-\rho)\bar{F}_-(x, \rho), F_-(x, \rho)A(-\rho) \\ &\quad + F_-(x, -\rho)B(-\rho) \rangle = 2i\rho A^*(-\rho)A(-\rho) - 2i\rho B^*(-\rho)B(-\rho) = 2iI_m. \end{aligned}$$

Thus, the relations (2.13) are proved.

The relations

$$\begin{aligned} \langle \bar{F}_+(x, \rho), F_-(x, \rho) \rangle &= \langle A^*(-\rho)\bar{F}_-(x, -\rho) + B^*(-\rho)\bar{F}_-(x, \rho), F_-(x, \rho) \rangle = 2i\rho A^*(-\rho), \\ \langle \bar{F}_+(x, \rho), F_-(x, -\rho) \rangle &= \langle \bar{F}_+(x, \rho), F_+(x, \rho)C(\rho) + F_+(x, -\rho)D(\rho) \rangle = 2i\rho D(\rho), \\ \langle \bar{F}_+(x, \rho), F_-(x, -\rho) \rangle &= \langle A^*(-\rho)\bar{F}_-(x, -\rho) + B^*(-\rho)\bar{F}_-(x, \rho), F_-(x, -\rho) \rangle = -2i\rho B^*(-\rho), \\ \langle \bar{F}_+(x, \rho), F_-(x, \rho) \rangle &= \langle \bar{F}_+(x, \rho), F_+(x, -\rho)C(-\rho) + F_+(x, \rho)D(-\rho) \rangle = 2i\rho C(-\rho) \end{aligned}$$

yield (2.14).

Using (2.11), (2.17), (2.9) and (2.10) again, we obtain

$$\begin{aligned} \langle \bar{F}_-(x, \rho), F_+(x, \rho) \rangle &= \langle \bar{F}_-(x, \rho), F_-(x, -\rho)A(\rho) + F_-(x, \rho)B(\rho) \rangle = -2i\rho A(\rho), \\ \langle \bar{F}_-(x, -\rho), F_+(x, \rho) \rangle &= \langle \bar{F}_-(x, -\rho), F_-(x, -\rho)A(\rho) + F_-(x, \rho)B(\rho) \rangle = 2i\rho B(\rho). \end{aligned}$$

Finally, we arrive at (2.15) and (2.16). □

The relation (2.15) gives the analytic continuation for $A(\rho)$ to the upper half-plane $\text{Im } \rho > 0$. Hence, the matrix functions $A(\rho)$ and $D(\rho) = A^*(-\bar{\rho})$ are analytic for $\text{Im } \rho > 0$ and $\rho A(\rho)$, $\rho D(\rho)$ are continuous for $\text{Im } \rho \geq 0$. The matrix functions $\rho B(\rho)$ and $\rho C(\rho)$ are continuous for real ρ .

Lemma 2.2. *The following asymptotic formulas are valid:*

$$\begin{aligned} A(\rho), D(\rho) &= I_m - \frac{\omega}{i\rho} + o(\rho^{-1}), \quad \omega = \frac{1}{2} \int_{-\infty}^{\infty} Q(t) dt \quad \text{Im } \rho \geq 0, \\ B(\rho), C(\rho) &= o(\rho^{-1}), \quad \rho \in \mathbb{R}, \end{aligned}$$

as $|\rho| \rightarrow \infty$.

Proof. The assertion of the lemma immediately follows from (2.14), (2.15), (2.16) and (2.3). \square

The matrix functions

$$S_-(\rho) = B(\rho)(A(\rho))^{-1}, \quad S_+(\rho) = C(\rho)(D(\rho))^{-1} \quad (2.18)$$

are called the left and the right *scattering matrices*, respectively. Denote

$$F_+^0(x, \rho) = F_+(x, \rho)(A(\rho))^{-1}, \quad F_-^0(x, \rho) = F_-(x, \rho)(D(\rho))^{-1}.$$

It follows from (2.11) and (2.12), that

$$F_+^0(x, \rho) = F_-(x, -\rho) + F_-(x, \rho)S_-(\rho), \quad F_-^0(x, \rho) = F_+(x, -\rho) + F_+(x, \rho)S_+(\rho).$$

Taking the asymptotic formulas (2.2) into account, we get

$$\begin{aligned} F_+^0(x, \rho) &\sim \exp(i\rho x)I_m + S_-(\rho) \exp(-i\rho x), \quad x \rightarrow -\infty, \\ F_+^0(x, \rho) &\sim \exp(i\rho x)T_+(\rho), \quad x \rightarrow +\infty, \\ F_-^0(x, \rho) &\sim \exp(-i\rho x)I_m + S_+(\rho) \exp(i\rho x), \quad x \rightarrow +\infty, \\ F_-^0(x, \rho) &\sim \exp(-i\rho x)T_-(\rho), \quad x \rightarrow -\infty. \end{aligned}$$

Thus, the matrix functions $S_{\pm}(\rho)$ generalize the scalar *reflection coefficients*, and the matrix functions $T_+(\rho) = (A(\rho))^{-1}$, $T_-(\rho) = (D(\rho))^{-1}$ generalize the *transmission coefficients* (see [1, Chapter 3]).

Lemma 2.3. *For real $\rho \neq 0$ the matrix functions $S_{\pm}(\rho)$ are continuous and the following relations hold:*

$$\begin{aligned} S_{\pm}^*(\rho) &= S_{\pm}(-\rho), \quad \|S_{\pm}(\rho)\| < 1, \quad S_{\pm}(\rho) = o(\rho^{-1}), \quad |\rho| \rightarrow \infty, \\ S_-^*(\rho)S_-(\rho) &= I_m - (A^*(\rho))^{-1}(A(\rho))^{-1}, \quad S_+^*(\rho)S_+(\rho) = I_m - (D^*(\rho))^{-1}(D(\rho))^{-1}, \end{aligned} \quad (2.19)$$

$$\lim_{\rho \rightarrow 0} \rho(S_-(\rho) + I_m)A(\rho) = 0_m, \quad \lim_{\rho \rightarrow 0} \rho(S_+(\rho) + I_m)D(\rho) = 0_m. \quad (2.20)$$

Proof. In view of (2.13), $A^*(\rho)A(\rho) > 0$, therefore $\det A(\rho) \neq 0$, $\det D(\rho) \neq 0$ for real $\rho \neq 0$. Hence the scattering matrices $S_{\pm}(\rho)$, defined by (2.18), are continuous functions for real $\rho \neq 0$. By virtue of (2.18) and (2.13),

$$S_-^*(\rho) = (A^*(\rho))^{-1}B^*(\rho) = B(-\rho)(A(-\rho))^{-1} = S_-(-\rho),$$

$$I_m = (A^*(\rho))^{-1}(I_m + B^*(\rho)B(\rho))(A(\rho))^{-1} = (A^*(\rho))^{-1}(A(\rho))^{-1} + S_-^*(\rho)S_-(\rho).$$

Similar results are valid for $S_+(\rho)$. The estimate $\|S_\pm(\rho)\| < 1$ follows from (2.19). Lemma 2.2 and (2.18) yield $S_\pm(\rho) = o(\rho^{-1})$, as $|\rho| \rightarrow \infty$.

Using (2.11), (2.12) and (2.18), we obtain

$$\begin{aligned}\rho F_+(x, \rho) &= (F_-(x, \rho)(S_-(\rho) + I_m) + F_-(x, -\rho) - F_-(x, \rho))\rho A(\rho), \\ \rho F_-(x, \rho) &= (F_+(x, \rho)(S_+(\rho) + I_m) + F_+(x, -\rho) - F_+(x, \rho))\rho D(\rho).\end{aligned}$$

Taking limits as $\rho \rightarrow 0$, we arrive at (2.20). \square

It follows from Lemma 2.3, that the matrices $S_\pm(\rho)$ belong to $L_2((-\infty, \infty); \mathbb{C}^{m \times m})$, so they have the Fourier transforms

$$R_\pm(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_\pm(\rho) \exp(\pm i\rho x) d\rho, \quad (2.21)$$

belonging to $L_2((-\infty, \infty); \mathbb{C}^{m \times m})$, $R_\pm(x) = R_\pm^*(x)$, and

$$S_\pm(\rho) = \int_{-\infty}^{\infty} R_\pm(x) \exp(\mp i\rho x) dx. \quad (2.22)$$

2.4. Eigenvalues. The values ρ^2 , for which the equation (1.1) has nonzero solutions $Y(x) \in L_2((-\infty, \infty); \mathbb{C}^m)$, are called *eigenvalues* of (1.1), and the corresponding solutions are called *eigenfunctions*.

Lemma 2.4. *There are no eigenvalues for $\rho^2 \geq 0$.*

Proof. Let $\rho_0^2 > 0$ be an eigenvalue and $Y_0(x)$ be a corresponding eigenfunction. Then $Y_0(x) = F_+(x, \rho_0)A_0 + F_+(x, -\rho_0)B_0$, where $A_0, B_0 \in \mathbb{C}^m$. But it follows from (2.2) and the relation $\lim_{x \rightarrow +\infty} Y_0(x) = 0$, that $A_0 = B_0 = 0$, so we arrive at the contradiction.

For $\rho = 0$ equation (1.1) has the solution $E(x) := F_+(x, 0) = I_m + o(1)$ as $x \rightarrow +\infty$. There exists a constant a , such that $\det E(x) \neq 0$ for $x \geq a$. One can easily check that the matrix function

$$Z(x) = E(x) \int_a^x (E^*(t)E(t))^{-1} dt$$

satisfies equation (1.1) for $\rho = 0$ and enjoys asymptotic representation $Z(x) = x(I_m + o(1))$ as $x \rightarrow +\infty$. Thus, the columns of the matrices $E(x)$ and $Z(x)$ form a fundamental system of solutions for equation (1.1) for $\rho = 0$. If equation (1.1) has the zero eigenvalue, then the corresponding eigenfunction admits the expansion $Y_0(x) = E(x)A_0 + Z(x)B_0$, $A_0, B_0 \in \mathbb{C}^m$. But in view of asymptotic behavior of $Y_0(x)$, $E(x)$ and $Z(x)$, we obtain $A_0 = B_0 = 0$. Hence, $\rho^2 = 0$ is not an eigenvalue of (1.1). \square

Let

$$\Lambda_+ := \{\lambda = \rho^2 : \operatorname{Im} \rho > 0, \det A(\rho) = 0\}.$$

Since $A(\rho)$ is analytic function in the upper half-plane and enjoys asymptotic representation $A(\rho) = I_m + O(\rho^{-1})$ as $|\rho| \rightarrow \infty$ by Lemma 2.2, the set Λ_+ is bounded and at most countable.

Lemma 2.5. *The set of eigenvalues coincide with Λ_+ . If $A(\rho_0)V_0 = 0$, $V_0 \in \mathbb{C}^m$, $V_0 \neq 0$, then $F_+(x, \rho_0)V_0$ is a vector eigenfunction, corresponding to the eigenvalue ρ_0^2 .*

Proof. 1. Let $\rho_0^2 \in \Lambda_+$. By virtue of (2.15), there exists a vector $V_0 \in \mathbb{C}^m$, such that

$$\langle \bar{F}_-(x, \rho_0), F_+(x, \rho_0) \rangle V_0 = 0.$$

Then the vector function $Y_0(x) := F_+(x, \rho_0)V_0$, belonging to $L_2((a, +\infty); \mathbb{C}^m)$ for every real a , satisfies the relation

$$\langle \bar{F}_-(x, \rho_0), Y_0(x) \rangle = \bar{F}'_-(x, \rho_0)Y_0(x) - \bar{F}_-(x, \rho_0)Y'_0(x) = 0.$$

Let x_0 be such that $\det F_-(x_0, \rho_0) \neq 0$ (such value exists in view of asymptotics (2.2)). Then the solution $Y_0(x)$ satisfies n linearly independent conditions:

$$Y'_0(x_0) - (\bar{F}_-(x_0, \rho_0))^{-1} \bar{F}'_-(x_0, \rho_0)Y_0(x_0) = 0.$$

The n linearly independent columns of the matrix $F_-(x, \rho_0)$ satisfy the same conditions by virtue of (2.9). Consequently, $Y_0(x) = F_-(x, \rho_0)U_0$, $U_0 \in \mathbb{C}^m$, so $Y_0(x) \in L_2((-\infty, a); \mathbb{C}^m)$. Thus, $Y_0(x)$ is an eigenfunction, and ρ_0^2 is an eigenvalue of equation (1.1).

2. On the contrary, let ρ_0^2 be an eigenvalue and $Y_0(x)$ be a corresponding eigenfunction. By virtue of the property (i_3) of the Jost solutions, $Y_0(x) = F_+(x, \rho_0)V_0 = F_-(x, \rho_0)U_0$, $V_0, U_0 \in \mathbb{C}^m$. On the one hand,

$$\langle \bar{F}_-(x, \rho_0), Y_0(x) \rangle = \langle \bar{F}_-(x, \rho_0), F_-(x, \rho_0) \rangle U_0 = 0.$$

On the other hand,

$$\langle \bar{F}_-(x, \rho_0), Y_0(x) \rangle = \langle \bar{F}_-(x, \rho_0), F_+(x, \rho_0) \rangle V_0 = 0.$$

In view of (2.15), $\det A(\rho_0) = 0$, i.e. $\rho_0^2 \in \Lambda_+$. □

The operator $-Y'' + Q(x)Y$ is self-adjoint in $L_2((-\infty, \infty); \mathbb{C}^m)$. Taking Lemma 2.4 into account, we conclude that eigenvalues ρ^2 of (1.1) are real and negative, and eigenfunctions, corresponding to different eigenvalues λ_k and λ_n are orthogonal:

$$\int_{-\infty}^{\infty} Y_k^*(x)Y_n(x) dx = 0.$$

In view of (2.14), the eigenvalues also coincide with the zeros of $\det D(\rho)$ in the upper half-plane.

Lemma 2.6. *The number of the eigenvalues is finite.*

Proof. Prove the assertion by contradiction. Suppose there is an infinite sequence $\{\rho_k^2\}_{k=1}^{\infty}$ of negative eigenvalues, and $\{Y_k(x)\}_{k=1}^{\infty}$ is an orthogonal sequence of corresponding vector eigenfunctions. Note that there can be multiple eigenvalues, they occur in the sequence $\{\rho_k^2\}_{k=1}^{\infty}$ multiple times with different eigenfunctions $Y_k(x)$. The eigenfunctions can be represented in the form

$$Y_k(x) = F_+(x, \rho_k)V_k = F_-(x, \rho_k)U_k, \quad \|V_k\| = \|U_k\| = 1,$$

Since $\rho_k = i\tau_k$, $\tau_k > 0$, and (2.8) holds, we get

$$Y_k^*(x) = V_k^* \bar{F}_+(x, \rho_k) = U_k^* \bar{F}_-(x, \rho_k).$$

Using the orthogonality of the eigenfunctions, we obtain for $k \neq n$

$$\begin{aligned} 0 &= \int_{-\infty}^{\infty} Y_k^*(x)Y_n(x) dx = V_k^* \int_a^{\infty} \bar{F}_+(x, \rho_k)F_+(x, \rho_n) dx V_n + U_k^* \int_{-\infty}^{-a} \bar{F}_-(x, \rho_k)F_-(x, \rho_n) dx U_n \\ &\quad + \int_{-a}^a Y_k^*(x)Y_k(x) dx + \int_{-a}^a Y_k^*(x)(Y_n(x) - Y_k(x)) dx =: \mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3 + \mathcal{I}_4. \end{aligned} \quad (2.23)$$

It follows from (2.2), that

$$F_+(x, \rho_n) = \exp(-\tau_n x)(I_m + \alpha_n(x)), \quad \bar{F}_+(x, \rho_k) = \exp(-\tau_k x)(I_m + \alpha_k^*(x)),$$

where $\|\alpha_k(x)\| \leq \frac{1}{8}$ as $x \geq a$ for all $k \geq 1$ and for sufficiently large a . Therefore

$$\begin{aligned} \mathcal{I}_1 &= V_k^* \int_a^\infty \exp(-(\tau_k + \tau_n)x)(I_m + \beta_{kn}(x)) dx V_k \\ &\quad + V_k^* \int_a^\infty \exp(-(\tau_k + \tau_n)x)(I_m + \beta_{kn}(x)) dx (V_n - V_k). \end{aligned}$$

Since the vectors V_k belong to the unit sphere, one can choose a convergent subsequence $\{V_{k_s}\}_{s=1}^\infty$. Further we consider V_k and V_n from such subsequence. Then for sufficiently large k and n we have

$$\begin{aligned} \left| V_k^* \int_a^\infty \exp(-(\tau_k + \tau_n)x)(I_m + \beta_{kn}(x)) dx (V_n - V_k) \right| \\ \leq \frac{3 \exp(-(\tau_k + \tau_n)a)}{2(\tau_k + \tau_n)} \|V_n - V_k\| \leq \frac{\exp(-(\tau_k + \tau_n)a)}{8(\tau_k + \tau_n)}. \end{aligned}$$

Hence

$$\mathcal{I}_1 \geq \frac{\exp(-(\tau_k + \tau_n)a)}{2(\tau_k + \tau_n)} \geq \frac{\exp(-2aT)}{4T}, \quad T := \max_k \tau_k.$$

Similar estimate is valid for \mathcal{I}_2 . Obviously, $\mathcal{I}_3 \geq 0$. Using the arguments, similar to the proof of [4, Theorem 2.3.4], one can show that $\mathcal{I}_4 \rightarrow 0$ as $k, n \rightarrow \infty$. Thus, for sufficiently large k and n , $\mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3 + \mathcal{I}_4 > 0$, that contradicts (2.23). Hence, the number of the eigenvalues is finite. \square

Further we denote the set of eigenvalues by $\{\rho_k^2\}_{k=1}^N$, $\rho_k = i\tau_k$, $\tau_k > 0$.

2.5 Scattering data.

Lemma 2.7. *The poles of the matrix function $(A(\rho))^{-1}$ in the upper half-plane $\text{Im } \rho > 0$ are simple.*

We will prove Lemma 2.7, using the following well-known fact (see [11, Lemma 2.2.1]).

Lemma 2.8. *If there do not exist two nonzero vectors a and b , such that*

$$\begin{aligned} A(\rho)a &= 0, \\ \frac{d}{d\rho}A(\rho)a + A(\rho)b &= 0, \end{aligned} \tag{2.24}$$

at some point ρ_0 , then ρ_0 is a simple pole of $(A(\rho))^{-1}$.

Proof of Lemma 2.7. Using (2.15), calculate the derivative

$$\begin{aligned} \frac{d}{d\rho}A(\rho) &= -\frac{d}{d\rho} \left(\frac{1}{2i\rho} \langle \bar{F}_-(x, \rho), F_+(x, \rho) \rangle \right) \\ &= -\frac{1}{2i\rho} \left(-\frac{1}{\rho} \langle \bar{F}_-(x, \rho), F_+(x, \rho) \rangle + \langle \frac{d}{d\rho} \bar{F}_-(x, \rho), F_+(x, \rho) \rangle + \langle \bar{F}_-(x, \rho), \frac{d}{d\rho} F_+(x, \rho) \rangle \right). \end{aligned}$$

Differentiate equation (2.6) for $\bar{F}_-(x, \rho)$ by ρ :

$$-\frac{d}{d\rho} \bar{F}_-''(x, \rho) + \frac{d}{d\rho} \bar{F}_-(x, \rho) Q(x) = 2\rho \bar{F}_-(x, \rho) + \rho^2 \frac{d}{d\rho} \bar{F}_-(x, \rho). \tag{2.25}$$

Using (1.1) and (2.25), one easily obtains

$$\frac{d}{dx} \left\langle \frac{d}{d\rho} \bar{F}_-(x, \rho), F_+(x, \rho) \right\rangle = -2\rho \bar{F}_-(x, \rho) F_+(x, \rho).$$

Hence

$$\left\langle \frac{d}{d\rho} \bar{F}_-(x, \rho), F_+(x, \rho) \right\rangle = \left\langle \frac{d}{d\rho} \bar{F}_-(x, \rho), F_+(x, \rho) \right\rangle_{x=-\infty} - 2\rho \int_{-\infty}^x \bar{F}_-(t, \rho) F_+(t, \rho) dt.$$

Similarly

$$\left\langle \bar{F}_-(x, \rho), \frac{d}{d\rho} F_+(x, \rho) \right\rangle = \left\langle \bar{F}_-(x, \rho), \frac{d}{d\rho} F_+(x, \rho) \right\rangle_{x=+\infty} + 2\rho \int_x^{\infty} \bar{F}_-(t, \rho) F_+(t, \rho) dt.$$

Finally, we get

$$\begin{aligned} \frac{d}{d\rho} A(\rho) &= -\frac{1}{\rho} A(\rho) - \frac{1}{2i\rho} \left(\left\langle \frac{d}{d\rho} \bar{F}_-(x, \rho), F_+(x, \rho) \right\rangle_{x=-\infty} \right. \\ &\quad \left. + \left\langle \bar{F}_-(x, \rho), \frac{d}{d\rho} F_+(x, \rho) \right\rangle_{x=+\infty} + 2\rho \int_{-\infty}^{\infty} \bar{F}_-(x, \rho) F_+(x, \rho) dx \right). \end{aligned} \quad (2.26)$$

By Lemma 2.5 the poles of $(A(\rho))^{-1}$ coincide with the eigenvalues of (1.1). Let ρ_k be one of the poles. Then $A(\rho_k)V_k = 0$ if and only if $F_+(x, \rho_k)V_k$ is a vector eigenfunction, corresponding to the eigenvalue ρ_k^2 . Note that

$$F_+(x, \rho_k)V_k = F_-(x, \rho_k)U_k, \quad V_k^* \bar{F}_+(x, \rho_k) = U_k^* \bar{F}_-(x, \rho_k), \quad U_k \in \mathbb{C}^m. \quad (2.27)$$

Using (2.26), (2.27) and (2.8), we derive

$$\begin{aligned} U_k^* \frac{d}{d\rho} A(\rho_k)V_k &= -\frac{1}{2i\rho_k} U_k^* \left\langle \frac{d}{d\rho} \bar{F}_-(x, \rho_k), F_-(x, \rho_k) \right\rangle_{x=-\infty} U_k \\ &\quad - \frac{1}{2i\rho_k} V_k^* \left\langle \bar{F}_+(x, \rho_k), \frac{d}{d\rho} F_+(x, \rho_k) \right\rangle_{x=+\infty} V_k + iU_k^* \int_{-\infty}^{\infty} F_-^*(x, \rho_k) F_-(x, \rho_k) dx U_k. \end{aligned}$$

Similarly to the scalar case (see [4, Theorem 3.4.1]), using integral transforms (2.4), one can show that for $\nu = 0, 1$

$$\begin{cases} \frac{d}{d\rho} \bar{F}_-^{(\nu)}(x, \rho_k) = O(1), & x \rightarrow -\infty, \\ \frac{d}{d\rho} F_+^{(\nu)}(x, \rho_k) = O(1), & x \rightarrow +\infty. \end{cases} \quad (2.28)$$

Applying these estimates together with (2.2), we obtain

$$\left\langle \frac{d}{d\rho} \bar{F}_-(x, \rho_k), F_-(x, \rho_k) \right\rangle_{x=-\infty} = 0_m, \quad \left\langle \bar{F}_+(x, \rho_k), \frac{d}{d\rho} F_+(x, \rho_k) \right\rangle_{x=+\infty} = 0_m.$$

Consequently,

$$U_k^* \frac{d}{d\rho} A(\rho_k)V_k = iU_k^* \int_{-\infty}^{\infty} F_-^*(x, \rho_k) F_-(x, \rho_k) dx U_k \neq 0.$$

Using (2.15), (2.27) and (2.9), we derive

$$U_k^* A(\rho_k) = -\frac{1}{2i\rho_k} U_k^* \left\langle \bar{F}_-(x, \rho_k), F_+(x, \rho_k) \right\rangle = -\frac{1}{2i\rho_k} V_k^* \left\langle \bar{F}_+(x, \rho_k), F_+(x, \rho_k) \right\rangle = 0.$$

Let $b \neq 0$ be some vector. Then

$$U_k^* \frac{d}{d\rho} A(\rho_k)V_k + U_k^* A(\rho_k)b \neq 0,$$

so (2.24) can not hold for nonzero a and b . By Lemma 2.8, ρ_k is a simple pole of $(A(\rho))^{-1}$. \square

Denote

$$R_k^- = \text{Res}_{\rho=\rho_k}(A(\rho))^{-1} = \lim_{\rho \rightarrow \rho_k} (\rho - \rho_k)(A(\rho))^{-1}, \quad R_k^+ = \text{Res}_{\rho=\rho_k}(D(\rho))^{-1} = \lim_{\rho \rightarrow \rho_k} (\rho - \rho_k)(D(\rho))^{-1}. \quad (2.29)$$

It follows from (2.29) and (2.14), that $R_k^- = -(R_k^+)^*$.

Clearly,

$$A(\rho_k)R_k^- = \lim_{\rho \rightarrow \rho_k} (\rho - \rho_k)A(\rho)(A(\rho))^{-1} = 0_m.$$

By Lemma 2.5, $F_+(x, \rho_k)R_k^-$ is an eigenfunction, so it can be represented in the form

$$F_+(x, \rho_k)R_k^- = iF_-(x, \rho_k)N_k^-. \quad (2.30)$$

Similarly

$$F_-(x, \rho_k)R_k^+ = iF_+(x, \rho_k)N_k^+. \quad (2.31)$$

We call the matrices N_k^- and N_k^+ the left and the right *weight matrices*, respectively.

Lemma 2.9. *The weight matrices have the following properties:*

$$\text{rank } N_k^+ = \text{rank } R_k^+ = \text{rank } R_k^- = \text{rank } N_k^-, \quad (2.32)$$

$$N_k^\pm = (N_k^\pm)^* \geq 0. \quad (2.33)$$

Proof. By virtue of (2.2), $\det F_+(x, \rho_k) \neq 0$ for $x > a$ and $\det F_-(x, \rho_k) \neq 0$ for $x < -a$, if a is sufficiently large. Therefore it follows from (2.30) and (2.31), that $\text{rank } R_k^\pm = \text{rank } N_k^\pm$. The relation $R_k^- = -(R_k^+)^*$ implies $\text{rank } R_k^+ = \text{rank } R_k^-$.

Now let us prove (2.33) for N_k^+ . The case of N_k^- is similar. One can easily show that

$$\frac{d}{dx} \langle \bar{F}_+(x, \rho_k), F_+(x, \rho) \rangle = (\rho^2 - \rho_k^2) \bar{F}_+(x, \rho_k) F_+(x, \rho).$$

Hence

$$\langle \bar{F}_+(t, \rho_k), F_+(t, \rho) \rangle \Big|_x^\infty = (\rho^2 - \rho_k^2) \int_x^\infty \bar{F}_+(t, \rho_k) F_+(t, \rho_k) dt.$$

Using (2.2), we obtain

$$\frac{1}{\rho^2 - \rho_k^2} i(N_k^+)^* \langle \bar{F}_+(x, \rho_k), F_+(x, \rho) \rangle iN_k^+ = (N_k^+)^* \int_x^\infty \bar{F}_+(x, \rho_k) F_+(x, \rho) dx N_k^+. \quad (2.34)$$

It follows from (2.8) and (2.31), that

$$i(N_k^+)^* \bar{F}_+(x, \rho_k) = -(R_k^+)^* \bar{F}_-(x, \rho_k) = R_k^- \bar{F}_-(x, \rho_k). \quad (2.35)$$

Using (2.34), (2.35), (2.15), (2.29), we derive

$$\begin{aligned} & \lim_{\rho \rightarrow \rho_k} \frac{1}{\rho^2 - \rho_k^2} i(N_k^+)^* \langle \bar{F}_+(x, \rho_k), F_+(x, \rho) \rangle iN_k^+ \\ &= \lim_{\rho \rightarrow \rho_k} \frac{1}{\rho^2 - \rho_k^2} (\rho - \rho_k)(A(\rho))^{-1} \langle \bar{F}_-(x, \rho), F_+(x, \rho) \rangle iN_k^+ \\ &+ R_k^- \lim_{\rho \rightarrow \rho_k} \frac{1}{\rho^2 - \rho_k^2} \langle \bar{F}_-(x, \rho_k) - \bar{F}_-(x, \rho), F_+(x, \rho) \rangle iN_k^+ \\ &= \frac{1}{2\rho_k} \lim_{\rho \rightarrow \rho_k} (A(\rho))^{-1} (-2i\rho) A(\rho) iN_k^+ - \frac{1}{2\rho_k} R_k^- \langle \frac{d}{d\rho} \bar{F}_-(x, \rho), F_+(x, \rho) \rangle_{\rho=\rho_k} iN_k^+ \end{aligned}$$

$$= N_k^+ - \frac{1}{2\rho_k} R_k^- \langle \frac{d}{d\rho} \bar{F}_-(x, \rho_k), F_-(x, \rho_k) \rangle R_k^+.$$

By virtue of (2.2) and (2.28),

$$\lim_{x \rightarrow -\infty} \langle \frac{d}{d\rho} \bar{F}_-(x, \rho_k), F_-(x, \rho_k) \rangle = 0_m.$$

Passing to the limit as $\rho \rightarrow \rho_k$, $x \rightarrow -\infty$ in (2.34), we arrive at the relation

$$N_k^+ = (N_k^+)^* \int_{-\infty}^{\infty} F_+^*(x, \rho_k) F_+(x, \rho_k) dx N_k^+.$$

Hence (2.33) is valid. \square

Remark 2.1. The ranks of the weight matrices coincide with the multiplicities of the corresponding eigenvalues, i.e. the number of corresponding linearly independent eigenfunctions. This fact can be proved similarly to [19, Lemma 4].

The collections

$$J_+ := \{\{S_+(\rho)\}_{\rho \in \mathbb{R}}, \{\rho_k^2, N_k^+\}_{k=1}^N\}, \quad J_- := \{\{S_-(\rho)\}_{\rho \in \mathbb{R}}, \{\rho_k^2, N_k^-\}_{k=1}^N\}$$

are called the *right* and the *left scattering data*, respectively.

Inverse scattering problem. *Given the scattering data J_+ (or J_-), construct the potential matrix Q .*

The next lemma establishes the connection between the left and the right scattering data.

Lemma 2.10. *The following relations hold*

$$\begin{aligned} S_-(\rho) &= -D^*(\rho) S_+^*(\rho) (D^*(-\rho))^{-1}, \quad \rho \in \mathbb{R} \setminus \{0\}, \\ N_k^- &= R_k^+ (N_k^+)^{-1} (R_k^+)^*, \quad k = \overline{1, N}. \end{aligned} \quad (2.36)$$

Since N_k^+ is not necessarily invertible, the matrix $(N_k^+)^{-1}$ is defined as follows:

$$\begin{aligned} N_k^+ &= U^* \mathcal{D} U, \quad U^* = U^{-1}, \quad \mathcal{D} = \text{diag}\{d_1, d_2, \dots, d_j, 0, \dots, 0\}, \quad d_l > 0, \quad l = \overline{1, j}, \\ (N_k^+)^{-1} &= U^* \mathcal{D}^{-1} U, \quad \mathcal{D}^{-1} = \text{diag}\{d_1^{-1}, d_2^{-1}, \dots, d_j^{-1}, 0, \dots, 0\}. \end{aligned}$$

Proof. Using (2.18) and (2.14), we derive

$$\begin{aligned} S_-(\rho) &= B(\rho) (A(\rho))^{-1} = -C^*(\rho) (D^*(-\rho))^{-1} = (S_+(\rho) D(\rho))^* (D^*(-\rho))^{-1} \\ &= -D^*(\rho) S_+^*(\rho) (D^*(-\rho))^{-1}. \end{aligned}$$

In view of (2.2), $\det F_-(x, \rho_k) \neq 0$ for $x < -a$, where a is sufficiently large. For such x , the relation (2.31) implies

$$R_k^+ = iC_k(x) N_k^+, \quad C_k(x) := (F_-(x, \rho_k))^{-1} F_+(x, \rho_k). \quad (2.37)$$

Consequently,

$$C_k(x) (N_k^+)^{1/2} = -i R_k^+ (N_k^+)^{-1/2}, \quad (2.38)$$

where

$$(N_k^+)^{-1/2} := U^* \mathcal{D}^{-1/2} U, \quad \mathcal{D}^{-1/2} = \text{diag}\{d_1^{-1/2}, d_2^{-1/2}, \dots, d_j^{-1/2}, 0, \dots, 0\},$$

due to the notation above. Using (2.30), (2.37) and (2.33), we obtain

$$N_k^- = -iC_k(x) R_k^- = iC_k(x) (R_k^+)^* = C_k(x) (N_k^+)^{1/2} ((N_k^+)^{1/2})^* C_k^*(x).$$

Applying (2.38), we arrive at (2.36). \square

2.6. Behavior of $(A(\rho))^{-1}$ in the neighborhood of $\rho = 0$

Lemma 2.11. *The following estimate is valid*

$$(A(\rho))^{-1} = O(1), \quad \rho \rightarrow 0, \quad \text{Im } \rho \geq 0. \quad (2.39)$$

Proof. Introduce the potentials

$$Q_r(x) = \begin{cases} Q(x), & |x| \leq r, \\ 0, & |x| > r, \end{cases} \quad r > 0.$$

Let $F_{\pm r}(x, \rho)$ be the corresponding Jost solutions. According to the integral equations (2.1), the matrix functions $F_{\pm r}(x, \rho)$ are entire in ρ for each fixed x , and

$$\lim_{r \rightarrow \infty} \sup_{\text{Im } \rho \geq 0} \sup_{\pm x \geq a} \|(F_{\pm r}^{(\nu)}(x, \rho) - F_{\pm}^{(\nu)}(x, \rho)) \exp(\mp i \rho x)\| = 0, \quad \nu = 0, 1, a \in \mathbb{R}.$$

Denote

$$A_r(\rho) := -\frac{1}{2i\rho} \langle \bar{F}_{-r}(x, \rho), F_{+r}(x, \rho) \rangle.$$

Obviously, the matrix function $\rho A_r(\rho)$ is entire in ρ and

$$\lim_{r \rightarrow \infty} \rho A_r(\rho) = \rho A(\rho) \quad (2.40)$$

uniformly for $\text{Im } \rho \geq 0$.

Let $(\rho_k^{(r)})^2$, $k = \overline{1, N^{(r)}}$, be the eigenvalues of the potential $Q_r(x)$ (counted with their multiplicities). Note that by virtue of Lemma 2.2 and (2.40),

$$T := \max_{k, r} |\rho_k^{(r)}| < \infty. \quad (2.41)$$

Let us show that $N^{(r)} \leq N_*$. Assume to the contrary that $N^{(r_i)} \rightarrow \infty$ as $i \rightarrow \infty$. Fix $\varepsilon > 0$. For each r , choose the largest possible subset of eigenvalues, such that $\|\rho_j^{(r)} - \rho_k^{(r)}\| < \varepsilon$, $j, k = \overline{1, M^{(r)}}$ (without loss of generality, we may assume that the eigenvalues with indices $k = \overline{1, M^{(r)}}$ belong to this subset). In view of (2.41), we have $M^{(r_i)} \rightarrow \infty$ as $i \rightarrow \infty$. The vector eigenfunctions, corresponding to the eigenvalues $\rho_k^{(r)}$, can be represented in the following form

$$Y_{kr}(x) = F_{+r}(x, \rho_k^{(r)}) V_k^{(r)} = F_{-r}(x, \rho_k^{(r)}) U_k^{(r)}, \quad V_k^{(r)}, U_k^{(r)} \in \mathbb{C}^m, \quad \|V_k^{(r)}\| = \|U_k^{(r)}\| = 1.$$

We assume that multiple eigenvalues are counted multiple times with pairwise orthogonal eigenfunctions. Since the unit sphere in R^m is compact, for sufficiently large r in the sequence $\{r_i\}$ one can choose a couple of eigenvalues (call them $\rho_1^{(r)}$ and $\rho_2^{(r)}$), such that

$$\|\rho_1^{(r)} - \rho_2^{(r)}\| < \varepsilon, \quad \|V_1^{(r)} - V_2^{(r)}\| < \varepsilon. \quad (2.42)$$

Following the proof of Lemma 2.6, we use the orthogonality of the eigenfunctions:

$$\begin{aligned} 0 &= \int_{-\infty}^{\infty} Y_{1r}^*(x) Y_{2r}(x) dx = V_1^{(r)*} \int_a^{\infty} \bar{F}_{+r}(x, \rho_1^{(r)}) F_{+r}(x, \rho_2^{(r)}) dx V_2^{(r)} \\ &+ U_1^{(r)*} \int_{-\infty}^{-a} \bar{F}_{-r}(x, \rho_1^{(r)}) F_{-r}(x, \rho_2^{(r)}) dx U_2^{(r)} + \int_{-a}^a Y_{1r}^*(x) Y_{1r}(x) dx + \int_{-a}^a Y_{1r}^*(x) (Y_{2r}(x) - Y_{1r}(x)) dx \\ &=: \mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3 + \mathcal{I}_4. \end{aligned} \quad (2.43)$$

Similarly to the proof of Lemma 2.6, one can show that

$$\mathcal{I}_1, \mathcal{I}_2 \geq \frac{\exp(-2aT)}{4T}, \quad \mathcal{I}_3 \geq 0,$$

where T was defined in (2.41). It follows from (2.42), that $\|\mathcal{I}_4\| \leq C\varepsilon$, where the constant C does not depend on r . Choosing sufficiently small ε , we obtain $\mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3 + \mathcal{I}_4 > 0$ and arrive at a contradiction with (2.43). Thus, $N^{(r)} \leq N_*$.

Denote

$$\tau^* := \frac{1}{2} \min \tau_k, \quad \mathcal{D} := \{\rho: |\rho| < \tau^*, \operatorname{Im} \rho > 0\}, \quad \mathcal{D}_\delta := \{\rho: \operatorname{Im} \rho > 0, \delta < |\rho| < \tau^*\},$$

$$P_r(\rho) := \prod_{\rho_k^{(r)} \in \mathcal{D}} \frac{\rho - \rho_k^{(r)}}{\rho + \rho_k^{(r)}}, \quad H_r(\rho) := P_r(\rho)(A_r(\rho))^{-1}.$$

By virtue of (2.40), the set $\{\rho_k^{(r)}\}$ does not have limit points in $\overline{\mathcal{D}}$ other than $\rho = 0$. Note that $\|(A_r(\rho))^{-1}\| \leq 1$ holds for real $\rho \neq 0$, since (2.19) holds for $A_r(\rho)$. Therefore, $(A_r(\rho))^{-1}$ is regular at $\rho = 0$, so it is bounded in $\partial\mathcal{D}$ for sufficiently large r . Since $N^{(r)} \leq N_*$ and $\rho_k^{(r)}$ in the product P_r tend to zero, $\|P_r(\rho)\| \leq C$ in $\partial\mathcal{D}$. Consequently, for sufficiently large r , $\|H_r(\rho)\| \leq C$ in $\partial\mathcal{D}$, where the constant C does not depend on r . The matrix function $H_r(\rho)$ is analytic in \mathcal{D} and continuous in $\overline{\mathcal{D}}$, therefore the estimate $\|H_r(\rho)\| \leq C$ holds in $\overline{\mathcal{D}}$.

It follows from (2.40), that

$$\lim_{r \rightarrow \infty} (A_r(\rho))^{-1} = (A(\rho))^{-1},$$

uniformly by $\rho \in \overline{\mathcal{D}}_\delta$. Clearly, $P_r(\rho) \rightarrow 1$ as $r \rightarrow \infty$. Hence $\lim_{r \rightarrow \infty} H_r(\rho) = (A(\rho))^{-1}$ uniformly in $\overline{\mathcal{D}}_\delta$. Consequently, $\|(A(\rho))^{-1}\| \leq C$ in $\overline{\mathcal{D}}_\delta$. Since C does not depend on δ , we arrive at (2.39). \square

Remark 2.2. A more detailed analysis of asymptotic behavior of the scattering matrix near $\rho = 0$ is presented in [20].

3. Derivation of the Gelfand-Levitan-Marchenko equation

In this section, we show the reduction of the studied inverse scattering problem to the linear integral equation. Although Gelfand-Levitan-Marchenko equation was used before (see, for example [12, 16]), we provide its derivation in order to make the paper self-contained.

Theorem 3.1. *For each fixed x , the matrix functions $K_\pm(x, t)$ (see (2.4)) satisfy the Gelfand-Levitan-Marchenko equations*

$$M_+(x+y) + K_+(x, y) + \int_x^\infty K_+(x, t)M_+(t+y) dt = 0_m, \quad y > x, \quad (3.1)$$

$$M_-(x+y) + K_-(x, y) + \int_{-\infty}^x K_-(x, t)M_-(t+y) dt = 0_m, \quad y < x, \quad (3.2)$$

where

$$M_\pm(x) = R_\pm(x) + \sum_{k=1}^N N_k^\pm \exp(\mp \tau_k x), \quad (3.3)$$

the matrix functions $R_\pm(x)$ are defined in (2.21), and $\rho_k = i\tau_k$.

Proof. The relation (2.12) can be rewritten in the form

$$F_-(x, \rho)((D(\rho))^{-1} - I_m) = F_+(x, \rho)S_+(\rho) + F_+(x, -\rho) - F_-(x, \rho). \quad (3.4)$$

Put $K_+(x, t) = 0_m$ for $t < x$ and $K_-(x, t) = 0_m$ for $t > x$. By virtue of (2.4) and (2.21)

$$\begin{aligned} F_+(x, \rho)S_+(\rho) + F_+(x, -\rho) - F_-(x, \rho) &= \left(\exp(i\rho x) + \int_{-\infty}^{\infty} K_+(x, t) \exp(i\rho t) dt \right) \\ &\cdot \left(\int_{-\infty}^{\infty} R_+(y) \exp(i\rho y) dy \right) + \int_{-\infty}^{\infty} (K_+(x, t) - K_-(x, t)) \exp(-i\rho t) dt \\ &=: \int_{-\infty}^{\infty} H(x, y) \exp(-i\rho y) dy. \end{aligned}$$

where

$$H(x, y) = K_+(x, y) - K_-(x, y) + R_+(x + y) + \int_x^{\infty} K_+(x, t) R_+(t + y) dt.$$

Thus, for each fixed x , the right-hand side of (3.4) is the Fourier transform of $H(x, y)$. Applying the inverse Fourier transform, we get

$$H(x, y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\rho) d\rho, \quad F(\rho) := F_-(x, \rho)((D(\rho))^{-1} - I_m) \exp(i\rho y). \quad (3.5)$$

Using (2.3) and Lemma 2.2, we obtain the following asymptotic formula for fixed $x < y$:

$$F(\rho) = \frac{1}{i\rho} \exp(i\rho(y - x))(\omega + o(1)), \quad |\rho| \rightarrow \infty, \operatorname{Im} \rho \geq 0. \quad (3.6)$$

The matrix function $F(\rho)$ is meromorphic in the upper-half plane with the poles ρ_k , $k = \overline{1, N}$. Let $C_{\delta, R}$ be a closed contour (with the counterclockwise circuit), which is the boundary of the domain $D_{\delta, R} := \{\rho : \operatorname{Im} \rho > 0, \delta < |\rho| < R\}$. The residue theorem yields

$$\frac{1}{2\pi i} \int_{C_{\delta, R}} F(\rho) d\rho = \sum_{k=1}^N \operatorname{Res}_{\rho=\rho_k} F(\rho).$$

It follows from the estimates (2.3), (2.39) and (3.6), that

$$\lim_{R \rightarrow \infty, \delta \rightarrow 0} \frac{1}{2\pi i} \int_{C_{\delta, R}} F(\rho) d\rho = \frac{1}{2\pi i} \int_{-\infty}^{\infty} F(\rho) d\rho.$$

Hence

$$H(x, y) = i \sum_{k=1}^N \operatorname{Res}_{\rho=\rho_k} F(\rho). \quad (3.7)$$

Using (2.31) and (2.4), we obtain

$$\begin{aligned} \operatorname{Res}_{\rho=\rho_k} F(\rho) &= F_-(x, \rho_k) R_k^+ \exp(i\rho_k y) = F_+(x, \rho_k) N_k^+ \exp(-\tau_k x) \\ &= N_k^+ \exp(-\tau_k(x + y)) + \int_x^{\infty} K_+(x, t) N_k^+ \exp(-\tau_k(t + y)) dt. \end{aligned} \quad (3.8)$$

Thus, combining (3.5), (3.7) and (3.8), and taking (3.3) into account, we arrive at equation (3.1). Equation (3.2) can be derived similarly. \square

Corollary 3.1. *The matrix functions $M_{\pm}(x)$ are absolutely continuous, and for each fixed $a > -\infty$ the following estimates are valid:*

$$\int_a^{\infty} \|M_{\pm}(\pm x)\| dx < \infty, \quad \int_a^{\infty} (1 + |x|) \|M'_{\pm}(\pm x)\| dx < \infty.$$

Proof. The proof is quite similar to the scalar case (see [1, Chapter 3], [4, Lemma 3.2.2]). \square

4. The unique solvability of the Gelfand-Levitan-Marchenko equation

The unique solvability of equations (3.1) and (3.2) plays a key role in the analysis of the inverse scattering problem.

Let us introduce the following conditions (Condition A_+ for the data J_+ and Condition A_- for the data J_-).

Condition A_{\pm} . For real $\rho \neq 0$, the matrix functions $S_{\pm}(\rho)$ are continuous, and

$$\|S_{\pm}(\rho)\| < 1, \quad S_{\pm}^*(\rho) = S_{\pm}(-\rho), \quad S_{\pm}(\rho) = o(\rho^{-1}), \quad |\rho| \rightarrow \infty.$$

The matrix functions

$$R_{\pm}(x) = \int_{-\infty}^{\infty} S_{\pm}(\rho) \exp(\pm i\rho x) d\rho$$

are absolutely continuous, $R_{\pm} \in L_2((-\infty, \infty); \mathbb{C}^{m \times m})$, $R_{\pm}(x) = R_{\pm}^*(x)$, and for each fixed $a > -\infty$ the following estimates hold

$$\int_a^{\infty} \|R_{\pm}(\pm x)\| dx < \infty, \quad \int_a^{\infty} (1 + |x|) \|R'_{\pm}(\pm x)\| dx < \infty. \quad (4.1)$$

Moreover, $\rho_k = i\tau_k$, $\tau_k > 0$, $N_k^{\pm} = (N_k^{\pm})^* \geq 0$, $k = \overline{1, N}$.

Theorem 4.2. *Let the collection $J_+ = \{S_+(\rho), \rho_k^2, N_k^+\}$ (or J_-) satisfies Condition A_+ (A_-). Then for each fixed x , equation (3.1) (respectively, (3.2)) has a unique solution $K_+(x, y) \in L((x, \infty); \mathbb{C}^{m \times m})$ (respectively, $K_-(x, y) \in L((-\infty, x); \mathbb{C}^{m \times m})$).*

Proof. One can easily show that for each fixed x , the operator

$$(J_x f)(y) = \int_x^{\infty} f(t) M_+(t + y) dt, \quad y > x,$$

is compact in the space of row-vector functions $L((x, \infty); \mathbb{C}^{m, T})$. So it is sufficient to prove that the homogeneous equation

$$f(y) + \int_x^{\infty} f(t) M_+(t + y) dt = 0 \quad (4.2)$$

has only zero solution. By virtue of (3.3), (4.1) and (4.2), the vector function $f(y)$ is bounded for $y > x$. Hence $f \in L_2((x, \infty); \mathbb{C}^{m, T})$. So we derive from (4.2):

$$\int_x^{\infty} f(y) f^*(y) dy + \int_x^{\infty} \int_x^{\infty} f(t) M_+(t + y) f^*(y) dt dy = 0.$$

Using (3.3), we obtain

$$\int_x^{\infty} f(y) f^*(y) dy + \sum_{k=1}^N \int_x^{\infty} \int_x^{\infty} f(t) \exp(-\tau_k t) N_k^+ \exp(-\tau_k y) f^*(y) dt dy$$

$$+ \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_x^{\infty} \int_x^{\infty} f(t) \exp(i\rho t) S_+(\rho) \exp(i\rho y) f^*(y) dt dy d\rho = 0. \quad (4.3)$$

Denote

$$\Phi(\rho) = \int_x^{\infty} f(t) \exp(i\rho t) dt.$$

By Parseval's identity,

$$\int_x^{\infty} f(y) f^*(y) dy = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Phi(\rho) \Phi^*(\rho) d\rho.$$

Then the relation (4.3) takes the form

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \Phi(\rho) (I_m + S(\rho)) \Phi^*(\rho) d\rho + \sum_{k=1}^N \Phi(\rho_k) N_k^+ (\Phi(\rho_k))^* = 0. \quad (4.4)$$

Let

$$H(\rho) = \frac{1}{2} (S_+(\rho) + S_+^*(\rho)).$$

Then

$$\operatorname{Re} \left\{ \int_{-\infty}^{\infty} \Phi(\rho) (I_m + S_+(\rho)) \Phi^*(\rho) d\rho \right\} = \int_{-\infty}^{\infty} \Phi(\rho) (I_m + H(\rho)) \Phi^*(\rho) d\rho.$$

It follows from $\|S_+(\rho)\| < 1$, that $\|H(\rho)\| < 1$, and, consequently, $I_m + H(\rho) > 0$ for real $\rho \neq 0$. Hence

$$\int_{-\infty}^{\infty} \Phi(\rho) (I_m + H(\rho)) \Phi^*(\rho) d\rho \geq 0.$$

Note that

$$\Phi(\rho_k) N_k^+ (\Phi(\rho_k))^* \geq 0.$$

Therefore, the equality (4.4) is possible only if

$$\int_{-\infty}^{\infty} \Phi(\rho) (I_m + H(\rho)) \Phi^*(\rho) d\rho = 0.$$

This relation yields $\Phi(\rho) \equiv 0$, so $f(y) \equiv 0$. Thus, the homogeneous equation (4.2) has a unique solution, and the unique solvability of (3.1) is proved. The proof for (3.2) is similar. \square

Note that the data J_+ and J_- in Theorem 4.2 are not required to be scattering data for some particular potential $Q(x)$. However, by virtue of Lemmas 2.3, 2.6, 2.9, Corollary 3.1 and other results of Section 2, Conditions A_{\pm} hold for the scattering data J_{\pm} of equation (1.1).

Corollary 4.2 (uniqueness theorem). *The potential Q in equation (1.1) is uniquely determined by the scattering data J_+ (or J_-).*

The following algorithm can be used for the solution of the inverse scattering problem.

Algorithm. Given the scattering data $J_+ = \{S_+(\rho), \rho_k^2, N_k^+\}$.

1. Construct $M_+(x)$ by formula (3.3).
2. Solve equation (3.1) and find $K_+(x, y)$.
3. Find the potential $Q(x)$ by (2.5).

The solution for J_- is similar.

Further we also need the following auxiliary fact.

Lemma 4.12. *Let the collections J_{\pm} satisfy Conditions A_{\pm} , and let $K_{\pm}(x, y)$ be the solutions of the integral equations (3.1) and (3.2). Define the matrix functions $F_{\pm}(x, \rho)$ by (2.4), and the matrix functions $Q_{\pm}(x)$ by (2.5). Then for each fixed $a > -\infty$*

$$\int_a^{\infty} (1 + |x|) \|Q_{\pm}(\pm x)\| dx < \infty,$$

and

$$-F_{\pm}''(x, \rho) + Q_{\pm}(x)F_{\pm}(x, \rho) = \rho^2 F_{\pm}(x, \rho).$$

Proof. The proof is quite technical and repeats the proof of [4, Lemma 3.3.1]. \square

5. Necessary and sufficient conditions

5.1. Main theorem. In this section, we formulate and prove the main result on the necessary and sufficient conditions for the solvability of the inverse scattering problem.

Note that Lemma 4.12, as in the scalar case, gives a “good” behavior of the potential on (a, ∞) , if we solve the Gelfand-Levitan-Marchenko equation by J_+ , and on $(-\infty, a)$, if we use J_- . Therefore we need the connection between the left and the right scattering data in the necessary and sufficient conditions. For definiteness, we formulate the following condition for $J_+ = \{S_+(\rho), \rho_k^2, N_k^+\}$.

Condition B. There exists a matrix function $D(\rho)$, such that:

1. $D(\rho)$ is analytical for $\text{Im } \rho > 0$ and $\rho D(\rho)$ is continuous for $\text{Im } \rho \geq 0$.
2. $\det D(\rho) = 0$ only for $\rho = \rho_k$, $k = \overline{1, N}$, and $\text{Res}_{\rho=\rho_k}(D(\rho))^{-1} = C_k N_k^+$ for some $C_k \in \mathbb{C}^{m \times m}$, $\det C_k \neq 0$.
3. $D(\rho) = I_m + O(\rho^{-1})$, as $|\rho| \rightarrow \infty$.
4. $(D(\rho))^{-1} = O(1)$, as $\rho \rightarrow 0$.
5. $(D^*(\rho))^{-1}(D(\rho))^{-1} = I_m - S_+^*(\rho)S_+(\rho)$ for real $\rho \neq 0$.
6. $\lim_{\rho \rightarrow 0} \rho(S_+(\rho) + I_m)D(\rho) = 0_m$, $\rho \in \mathbb{R}$.
7. Define

$$S_-(\rho) = -D^*(\rho)S_+^*(\rho)(D^*(-\rho))^{-1} \quad (5.1)$$

and $R_-(x)$ by formula (2.21). The matrix function $R_-(x)$ is absolutely continuous and for each fixed $a > -\infty$

$$\int_{-\infty}^a \|R_-(x)\| dx < \infty, \quad \int_{-\infty}^a (1 + |x|) \|R'_-(x)\| dx < \infty.$$

Remark 5.3. In the scalar case ($m = 1$) the function $D(\rho)$, satisfying Conditions B1–B5, is uniquely determined by the following formula ([1, Theorem 3.5.1], [4, Theorem 3.3.1]):

$$D(\rho) = \prod_{k=1}^N \frac{\rho - i\tau_k}{\rho + i\tau_k} \exp(\gamma(\rho)), \quad \gamma(\rho) = -\frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\ln(1 - |S_+(\xi)|^2)}{\xi - \rho} d\xi, \quad \text{Im } \rho > 0.$$

Theorem 5.3 (Necessary and sufficient conditions). *For data $J_+ = \{S_+(\rho)\}_{\rho \in \mathbb{R}}, \{\rho_k^2, N_k^+\}_{k=1}^N\}$ to be the right scattering data for a certain potential $Q = Q^*$, satisfying (1.2), it is necessary and sufficient to satisfy Condition A_+ and Condition B.*

The necessity part of Theorem 5.3 was proved in Sections 2 and 4. Let us prove the sufficiency part. Let the data J_+ satisfy Condition A_+ and Condition B. Construct $A(\rho) := D^*(-\bar{\rho})$, $R_k^+ = \text{Res}_{\rho=\rho_k}(D(\rho))^{-1}$, $R_k^- := \text{Res}_{\rho=\rho_k}(A(\rho))^{-1} = -(R_k^+)^*$ and N_k^- by formula (2.36).

Lemma 5.13. *The data $J_- = \{S_-(\rho), \rho_k^2, N_k^-\}$, constructed above, satisfy Condition A_- .*

Proof. It follows from (5.1), Conditions B1, B3 and A_+ , that the matrix function $S_-(\rho)$ is continuous for real $\rho \neq 0$ and $S_-(\rho) = o(\rho^{-1})$ as $|\rho| \rightarrow \infty$.

Using the relation $S_+^*(\rho) = S_+(-\rho)$ and Condition B5, we derive

$$(I_m - S_+(\rho)S_+^*(\rho))S_+(\rho) = S_+(\rho)(I_m - S_+^*(\rho)S_+(\rho)),$$

$$(D^*(-\rho))^{-1}(D(-\rho))^{-1}S_+(\rho) = S_+^*(-\rho)(D^*(\rho))^{-1}(D(\rho))^{-1}.$$

Taking (5.1) into account, we obtain

$$S_-^*(\rho) = -(D(-\rho))^{-1}S_+(\rho)D(\rho) = -D^*(-\rho)S_+^*(-\rho)(D^*(\rho))^{-1} = S_-(-\rho), \quad (5.2)$$

for real $\rho \neq 0$.

Furthermore,

$$\begin{aligned} S_-^*(\rho)S_-(\rho) &= D^*(-\rho)S_+^*(-\rho)S_+(-\rho)(D^*(-\rho))^{-1} \\ &= D^*(-\rho) \{I_m - (D^*(-\rho))^{-1}(D(-\rho))^{-1}\} (D^*(-\rho))^{-1} = I_m - (D(-\rho))^{-1}(D^*(-\rho))^{-1}. \end{aligned}$$

Since $A(\rho) := D^*(-\bar{\rho})$, we have

$$S_-^*(\rho)S_-(\rho) = I_m - (A^*(\rho))^{-1}(A(\rho))^{-1}.$$

Consequently, $\|S_-(\rho)\| < 1$ for real $\rho \neq 0$.

Since $S_-(\rho) \in L_2((-\infty, \infty); \mathbb{C}^{m \times m})$, the Fourier transform $R_-(x)$ also belongs to $L_2((-\infty, \infty); \mathbb{C}^{m \times m})$. The relation $R_-^*(x) = R_-(x)$ follows from (5.2). The estimates (4.1) follow from Condition B7.

Finally, $(N_k^-)^* = N_k^- \geq 0$ follows from (2.36) and Condition A_+ . \square

Thus, the data J_{\pm} satisfy Condition A_{\pm} . By Theorem 4.2, the Gelfand-Levitan-Marchenko equations (3.1), (3.2) have unique solutions $K_{\pm}(x, y)$. Then one can construct matrix functions $F_{\pm}(x, \rho)$ by (2.4) and $Q_{\pm}(x)$ by (2.5), satisfying the assertion of Lemma 4.12. Our next goal is to prove that $Q_+(x) \equiv Q_-(x)$ and J_{\pm} are their scattering data.

Lemma 5.14. *The following relations hold*

$$F_-(x, \rho)S_-(\rho) + F_-(x, -\rho) = F_+(x, \rho)(A(\rho))^{-1}, \quad (5.3)$$

$$F_+(x, \rho)S_+(\rho) + F_+(x, -\rho) = F_-(x, \rho)(D(\rho))^{-1}, \quad (5.4)$$

$$F_+(x, \rho_k)R_k^- = F_-(x, \rho_k)iN_k^-, \quad F_-(x, \rho_k)R_k^+ = F_+(x, \rho_k)iN_k^+, \quad k = \overline{1, N}, \quad (5.5)$$

for $x < -a$ and $x > a$, where a is sufficiently large.

Proof. Divide the proof into three steps.

Step 1. Denote

$$\Phi(x, y) = R_+(x + y) + \int_x^\infty K_+(x, t)R_+(t + y) dt.$$

For each fixed x , $\Phi(x, y) \in L_2((-\infty, \infty); \mathbb{C}^{m \times m})$. It follows from (2.4) and (2.22), that

$$\begin{aligned}
F_+(x, \rho)S_+(\rho) &= \left(\exp(i\rho x) I_m + \int_x^\infty K_+(x, t) \exp(i\rho t) dt \right) \cdot \int_{-\infty}^\infty R_+(\xi) \exp(-i\rho\xi) d\xi \\
&= \int_{-\infty}^\infty R_+(x+y) \exp(-i\rho y) dy + \int_x^\infty K_+(x, t) \int_{-\infty}^\infty R_+(t+y) \exp(-i\rho y) dy dt \\
&= \int_{-\infty}^\infty \Phi(x, y) \exp(-i\rho y) dy. \quad (5.6)
\end{aligned}$$

On the other hand, (3.1), (3.3) and (2.4) imply

$$\Phi(x, y) = -K_+(x, y) - \sum_{k=1}^N \exp(-\tau_k y) F_+(x, \rho_k) N_k^+, \quad x < y.$$

Therefore

$$\begin{aligned}
\int_{-\infty}^\infty \Phi(x, y) \exp(-i\rho y) dy &= \int_{-\infty}^x \Phi(x, y) \exp(-i\rho y) dy + \exp(-i\rho x) I_m - F_+(x, -\rho) \\
&\quad - \sum_{k=1}^N \exp(-i\rho x) \frac{\exp(-\tau_k x)}{\tau_k + i\rho} F_+(x, \rho_k) N_k^+. \quad (5.7)
\end{aligned}$$

Combining (5.6) and (5.7), we obtain

$$F_+(x, \rho)S_+(\rho) + F_+(x, -\rho) = H_-(x, \rho)(D(\rho))^{-1}, \quad (5.8)$$

where

$$\begin{aligned}
H_-(x, \rho) &= \left[\int_{-\infty}^x \Phi(x, y) \exp(-i\rho y) dy + \exp(-i\rho x) I_m \right. \\
&\quad \left. - \sum_{k=1}^N \frac{\exp(-(i\rho + \tau_k)x)}{\tau_k + i\rho} F_+(x, \rho_k) N_k^+ \right] D(\rho). \quad (5.9)
\end{aligned}$$

Rewrite (5.8) in the form

$$H_-(x, \rho) = (F_+(x, \rho)S_+(\rho) + F_+(x, -\rho))D(\rho).$$

Clearly, $H_-(x, \rho)$ is continuous for real $\rho \neq 0$. Moreover, it follows from Condition B6, that

$$\lim_{\rho \rightarrow 0} \rho H_-(x, \rho) = 0_m \quad (5.10)$$

for each fixed x , so $\rho H_-(x, \rho)$ is continuous for $\rho \in \mathbb{R}$.

In view of (5.9), the matrix function $H_-(x, \rho)$ is analytic in the upper half-plane, except for the poles $\rho = \rho_k$. But

$$\text{Res}_{\rho=\rho_k} H_-(x, \rho) = iF_+(x, \rho_k)N_k^+ D(\rho_k).$$

According to Condition B2, $\text{Res}_{\rho=\rho_k} (D(\rho))^{-1} = C_k N_k^+$, so $N_k^+ D(\rho_k) = 0_m$. Hence, $\rho H_-(x, \rho)$ is analytic for $\text{Im } \rho > 0$ and continuous for $\text{Im } \rho \geq 0$, and (5.10) is valid for $\text{Im } \rho \geq 0$.

Using (5.9), we get

$$\text{Res}_{\rho=\rho_k} H_-(x, \rho)(D(\rho))^{-1} = H_-(x, \rho_k)R_k^+ = F_+(x, \rho_k)iN_k^+, \quad (5.11)$$

$$\lim_{|\rho| \rightarrow \infty} H_-(x, \rho) \exp(i\rho x) = I_m, \quad \text{Im } \rho \geq 0. \quad (5.12)$$

Similarly one can show that

$$F_-(x, \rho)S_-(\rho) + F_-(x, -\rho) = H_+(x, \rho)(A(\rho))^{-1}, \quad (5.13)$$

where the matrix function $H_+(x, \rho)$ is analytic for $\text{Im } \rho > 0$, $\rho H(x, \rho)$ is continuous for $\text{Im } \rho \geq 0$, and

$$H_+(x, \rho_k)R_k^- = F_-(x, \rho_k)iN_k^-, \quad (5.14)$$

$$\lim_{|\rho| \rightarrow \infty} H_+(x, \rho) \exp(-i\rho x) = I_m, \quad \lim_{\rho \rightarrow 0} \rho H_+(x, \rho) = 0_m, \quad \text{Im } \rho \geq 0. \quad (5.15)$$

Note that for the proof, one needs the relation

$$\lim_{\rho \rightarrow 0} \rho(S_-(\rho) - I_m)A(\rho) = 0,$$

which can be easily derived from Condition B6.

Step 2. It follows from (5.8), that

$$F_+(x, -\rho)S_+(-\rho) + F_+(x, \rho) = H_-(x, -\rho)(D(-\rho))^{-1}. \quad (5.16)$$

Substitute $F_+(x, -\rho)$ from (5.8) into (5.16):

$$F_+(x, \rho)(I_m - S_+(\rho)S_+(-\rho)) = H_-(x, -\rho)(D(-\rho))^{-1} - H_-(x, \rho)(D(\rho))^{-1}S_+(-\rho)$$

Using Condition B6, (5.2) and the relation $A(\rho) = D^*(-\rho)$, we obtain

$$F_+(x, \rho)(A(\rho))^{-1} = H_-(x, -\rho) + H_-(x, \rho)S_-(\rho). \quad (5.17)$$

Put $\bar{H}_+(x, \rho) := H_+^*(x, -\rho)$. Taking (2.8) and (5.2) into account, we obtain from (5.13):

$$(D(\rho))^{-1}\bar{H}_+(x, \rho) = \bar{F}_-(x, -\rho) + S_-(\rho)\bar{F}_-(x, \rho). \quad (5.18)$$

It follows from (5.17) and (5.18), that

$$\begin{aligned} H_-(x, -\rho)\bar{F}_-(x, \rho) - H_-(x, \rho)\bar{F}_-(x, -\rho) \\ = F_+(x, \rho)(A(\rho))^{-1}\bar{F}_-(x, \rho) - H_-(x, \rho)(D(\rho))^{-1}\bar{H}_+(x, \rho) =: G(x, \rho). \end{aligned} \quad (5.19)$$

For each fixed x , the matrix function $G(x, \rho)$ is meromorphic in the upper half-plane with the simple poles ρ_k . Using (5.19), we derive

$$\text{Res}_{\rho=\rho_k} G(x, \rho_k) = F_+(x, \rho_k)R_k^- \bar{F}_-(x, \rho_k) - H_-(x, \rho_k)R_k^+ \bar{H}_+(x, \rho_k). \quad (5.20)$$

The relation (5.14) yields

$$R_k^+ \bar{H}_+(x, \rho_k) = iN_k^- \bar{F}_-(x, \rho_k). \quad (5.21)$$

Multiplying the relation (5.11) by $i(N_k^+)^{-1}(R_k^+)^*$ ($(N_k^+)^{-1}$ was defined in Lemma 2.10) and using (2.36), we obtain

$$F_+(x, \rho_k)R_k^- = H_-(x, \rho_k)iN_k^-. \quad (5.22)$$

Substituting (5.21) and (5.22) into (5.20), we get

$$\text{Res}_{\rho=\rho_k} G(x, \rho_k) = 0_m,$$

so the matrix function $G(x, \rho)$ is analytic for $\text{Im } \rho > 0$. According to (5.19), (5.10), (5.12), (5.15), (2.4) and Conditions B1, B4, B5, the matrix function $\rho G(x, \rho)$ is continuous for $\text{Im } \rho \geq 0$ and

$$\lim_{\rho \rightarrow 0} \rho G(x, \rho) = 0_m, \quad (5.23)$$

$$\lim_{|\rho| \rightarrow \infty} G(x, \rho) = 0_m, \quad (5.24)$$

where $\text{Im } \rho \geq 0$.

It follows from (5.19), that

$$G(x, \rho) = -G(x, -\rho) \quad (5.25)$$

for real $\rho \neq 0$. One can continue the matrix function $G(x, \rho)$ to the lower half-plane by formula (5.25). Then $G(x, \rho)$ is analytic in $\mathbb{C} \setminus \{0\}$. By virtue of (5.23), the point $\rho = 0$ is a removable singularity, and $G(x, \rho)$ is entire in ρ . The relation (5.24) holds for all ρ , consequently, by Liouville's theorem, $G(x, \rho) \equiv 0_m$, i.e.

$$H_-(x, -\rho) \bar{F}_-(x, \rho) = H_-(x, \rho) \bar{F}_-(x, -\rho), \quad \rho \in \mathbb{R}. \quad (5.26)$$

Step 3. Similarly to Step 2, starting from (5.13) and (5.18), one can derive

$$\begin{aligned} F_-(x, -\rho) \bar{F}_-(x, \rho) - F_-(x, \rho) \bar{F}_-(x, -\rho) &= H_+(x, \rho) (A(\rho))^{-1} \bar{F}_-(x, \rho) \\ &\quad - F_-(x, \rho) (D(\rho))^{-1} \bar{H}_+(x, \rho) =: \tilde{G}(x, \rho). \end{aligned}$$

One can show that $\tilde{G}(x, \rho) \equiv 0_m$ similarly to $G(x, \rho) \equiv 0_m$. Consequently,

$$F_-(x, -\rho) \bar{F}_-(x, \rho) = F_-(x, \rho) \bar{F}_-(x, -\rho), \quad \rho \in \mathbb{R}. \quad (5.27)$$

The relations (5.26) and (5.27) together yield

$$P(x, \rho) = P(x, -\rho), \quad P(x, \rho) := H_-(x, \rho) (F_-(x, \rho))^{-1}, \quad \rho \in \mathbb{R},$$

for x such that $\det F_-(x, \pm \rho) \neq 0$. It follows from the properties of the Jost solution and (5.12), that for $x < -a$ (where a is sufficiently large), $P(x, \rho)$ is analytic for $\text{Im } \rho > 0$ and continuous for $\text{Im } \rho \geq 0$, $\rho \neq 0$, and $\lim_{|\rho| \rightarrow \infty} P(x, \rho) = I_m$. We continue the matrix function $P(x, \rho)$ to the half-plane $\text{Im } \rho < 0$ by the formula $P(x, \rho) = P(x, -\rho)$. In view of (5.10), the singularity of $P(x, \rho)$ is removable at $\rho = 0$. Thus, the matrix function $P(x, \rho)$ is entire and bounded. By Liouville's theorem, $P(x, \rho) \equiv I_m$, i.e. $H_-(x, \rho) \equiv F_-(x, \rho)$ for $x < -a$. Now the relations (5.3), (5.4), (5.5) follow from (5.17), (5.8), (5.11) and (5.22) for $x < -a$. Symmetrically one can prove them for $x > a$, if a is sufficiently large. □

Let us finish the proof of Theorem 5.3. It follows from (5.4), that

$$-F_-''(x, \rho) + Q_+(x) F_-(x, \rho) = \rho^2 F_-(x, \rho)$$

for $x < -a$. Consequently, $Q_+(x) = Q_-(x)$ for $x < -a$, the potential $Q_+(x)$ satisfies the condition (1.2), and the Jost solution $\tilde{F}_-(x, \rho)$ for $Q_+(x)$ equals $F_-(x, \rho)$ for $x < -a$. Construct for $Q_+(x)$ the matrix functions

$$\tilde{A}(\rho) = -\frac{1}{2i\rho} \langle \bar{F}_-(x, \rho), F_+(x, \rho) \rangle, \quad \tilde{B}(\rho) = \frac{1}{2i\rho} \langle \bar{F}_-(x, -\rho), F_+(x, \rho) \rangle, \quad x < -a.$$

Then for $x < -a$ the relation (2.11) holds:

$$F_+(x, \rho) = F_-(x, -\rho) \tilde{A}(\rho) + F_-(x, \rho) \tilde{B}(\rho).$$

Comparing this relation with (5.3), we obtain

$$\tilde{A}(\rho) = A(\rho), \quad \tilde{B}(\rho) = S_-(\rho) A(\rho).$$

Using these relations and (5.5), we conclude that $J_- = \{S_-(\rho), \rho_k, N_k^-\}$ are the left scattering data for the potential $Q_+(x)$. Similarly, using (5.4), we prove that $J_+ = \{S_+(\rho), \rho_k, N_k^+\}$ are the right scattering data for $Q_+(x)$. Symmetrically one can use the relations of Lemma 5.14 to prove that J_- and J_+ are the scattering data for the potential $Q_-(x)$. By virtue of the uniqueness theorem (Corollary 4.2), $Q_+(x) \equiv Q_-(x)$ and J_+, J_- are the scattering data for this potential. Theorem 5.3 is proved.

5.2. Reflectionless potentials. Let us consider the case $S(\rho) \equiv 0_m$.

Theorem 5.4. *For the data $S_+(\rho) \equiv 0_m$, ρ_k and N_k^+ , $k = \overline{1, N}$, to be the right scattering data for a certain potential $Q = Q^*$, satisfying (1.2), it is necessary and sufficient to satisfy the following conditions: $\rho_k = i\tau_k$, $\tau_k > 0$, numbers ρ_k are distinct, $N_k^+ = (N_k^+)^* \geq 0$.*

Proof. The necessity part follows directly from Theorem 5.3. In order to prove the sufficiency part, we have to check Conditions A₊ and B. It is sufficient to construct the matrix function $U(\rho)$, meromorphic with the simple poles ρ_k and zeros $-\rho_k$, such that

$$\operatorname{Res}_{\rho=\rho_k} U(\rho) = C_k N_k^+, \quad k = \overline{1, N}, \det C_k \neq 0, \quad (5.28)$$

$$U(\rho) = I_m + O(\rho^{-1}), \quad |\rho| \rightarrow \infty, \quad U^*(\rho)U(\rho) = I_m, \quad \rho \in \mathbb{R}. \quad (5.29)$$

Following [6], let us find such function in the form

$$U(\rho) = \left(I_m + \frac{2\rho_N}{\rho - \rho_N} P_N\right) \dots \left(I_m + \frac{2\rho_2}{\rho - \rho_2} P_2\right) \left(I_m + \frac{2\rho_1}{\rho - \rho_1} P_1\right),$$

where P_k are orthogonal projectors. Note that $U(\rho)$ does not have any other poles and zeros except for ρ_k and $-\rho_k$, $k = \overline{1, N}$, respectively, and fulfills (5.29). It remains to choose the projectors P_k to fulfill (5.28). For $k = 1$ we have

$$\operatorname{Res}_{\rho=\rho_1} U(\rho) = \left(I_m + \frac{2\rho_N}{\rho_1 - \rho_N} P_N\right) \dots \left(I_m + \frac{2\rho_2}{\rho_1 - \rho_2} P_2\right) 2\rho_1 P_1 = C_1 N_1^+.$$

Consequently, $\operatorname{Ker} P_1 = \operatorname{Ker} N_1^+$, so take such P_1 , that $I_m - P_1$ is an orthogonal projector to $\operatorname{Ker} N_1^+$. Further,

$$\operatorname{Res}_{\rho=\rho_2} U(\rho) = \left(I_m + \frac{2\rho_N}{\rho_2 - \rho_N} P_N\right) \dots 2\rho_2 P_2 \left(I_m + \frac{2\rho_1}{\rho_2 - \rho_1} P_1\right) = C_2 N_2^+.$$

We have $\operatorname{Ker} P_2 = \operatorname{Ker} N_2^+(I + \frac{2\rho_1}{\rho_2 - \rho_1} P_1)^{-1}$, so make $I_m - P_2$ to be a projector to the space $(I_m + \frac{2\rho_1}{\rho_2 - \rho_1} P_1) \operatorname{Ker} N_2^+$ and continue the process for $k = 3, \dots, N$. Clearly, the matrix function $D(\rho) = (U(\rho))^{-1}$ satisfies Condition B, so the sufficiency part is proved. \square

Remark 5.4. In the general case, we can search for the matrix function $D(\rho)$ in the form $D(\rho) = (U(\rho))^{-1} H(\rho)$, where $U(\rho)$ is the matrix function, constructed above by the discrete scattering data, and $H(\rho)$ satisfies Condition B with $\det H(\rho) \neq 0$, $\operatorname{Im} \rho \geq 0$, $\rho \neq 0$ instead of B2 and

$$(H^*(\rho))^{-1} (H(\rho))^{-1} = U(\rho) (I - S_+^*(\rho) S_+(\rho)) U^*(\rho)$$

instead of B5.

5.3. Riemann problem and application to the matrix KdV equation We note that Condition B is related to the following matrix Riemann problem.

Riemann problem (RP). *Given a continuous matrix function $G(\rho)$, $\rho \in \mathbb{R} \setminus \{0\}$, distinct numbers ρ_k , $k = \overline{1, N}$, $\operatorname{Im} \rho_k > 0$, and matrices N_k^+ . Find a matrix function $D(\rho)$ with the following properties:*

1. $D(\rho)$ is analytical for $\text{Im } \rho > 0$ and $\rho D(\rho)$ is continuous for $\text{Im } \rho \geq 0$.
2. $\det D(\rho) = 0$ only for $\rho = \rho_k$, $k = \overline{1, N}$, and $\text{Res}_{\rho=\rho_k}(D(\rho))^{-1} = C_k N_k^+$ for some $C_k \in \mathbb{C}^{m \times m}$, $\det C_k \neq 0$.
3. $D(\rho) = I_m + O(\rho^{-1})$, as $|\rho| \rightarrow \infty$.
4. $(D(\rho))^{-1} = O(1)$, as $\rho \rightarrow 0$.
5. $(D^*(\rho))^{-1}(D(\rho))^{-1} = G(\rho)$ for real $\rho \neq 0$.

There is an extensive literature devoted to the Riemann (or Riemann-Hilbert) problem (see, for example, [21, 22, 23]). However, the formulated problem RP differs from the classical case by the singularity at $\rho = 0$. Here we prove the uniqueness theorem for RP.

Theorem 5.5. *If RP has a solution, this solution is unique.*

Proof. Let, on the contrary, RP have two solutions $D_1(\rho)$ and $D_2(\rho)$. Consider the matrix function $U(\rho) := (D_2(\rho))^{-1}D_1(\rho)$. Clearly, this function is analytical for $\text{Im } \rho > 0$ and continuous for $\text{Im } \rho \geq 0$, $\rho \neq 0$. Moreover,

$$U^*(\rho)U(\rho) = I_m, \quad \rho \in \mathbb{R} \setminus \{0\}, \quad (5.30)$$

$$U(\rho) = I_m + O(\rho^{-1}), \quad |\rho| \rightarrow \infty, \quad U(\rho) = O(\rho^{-1}), \quad \rho \rightarrow 0.$$

The matrix function $V(\rho) = U(\frac{1}{\rho})$ is analytical in the lower half-plane and continuous for $\text{Im } \rho \leq 0$, $V(0) = I_m$ and $V(\rho) = O(\rho)$ as $|\rho| \rightarrow \infty$. By virtue of (5.30), the matrix function $V(\rho)$ is bounded for real ρ . It follows from the Phragmen-Lindelöf theorem [24], that $V(\rho) = O(1)$ as $|\rho| \rightarrow \infty$ in the lower half-plane. Hence $U(\rho) = O(1)$ as $\rho \rightarrow 0$.

Consider the closed contour $C_{\delta, R}$, being the boundary of the region $\{\rho: \text{Im } \rho \geq 0, \delta < \rho < R\}$. Cauchy's integral formula yields

$$U(\rho) - I_m = \frac{1}{2\pi i} \int_{C_{\delta, R}} \frac{U(\xi) - I_m}{\xi - \rho} d\xi, \quad \text{Im } \rho > 0.$$

Passing to the limit as $R \rightarrow \infty$ and $\delta \rightarrow 0$ and taking the behavior of $U(\rho)$ at infinity and zero into account, we obtain

$$U(\rho) - I_m = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{U(\xi) - I_m}{\xi - \rho} d\xi, \quad \text{Im } \rho > 0. \quad (5.31)$$

Here and below when necessary, the integral is considered in the principle value sense. For $\rho \in \mathbb{R} \setminus \{0\}$, we derive

$$U(\rho) - I_m = \frac{1}{\pi i} \int_{-\infty}^{\infty} \frac{U(\xi) - I_m}{\xi - \rho} d\xi. \quad (5.32)$$

Note that $\det U(\rho) \neq 0$ for $\text{Im } \rho \geq 0$, $\rho \neq 0$, and the matrix function $(U(\rho))^{-1}$ has the similar properties as $U(\rho)$. So we obtain the formula

$$(U(\rho))^{-1} - I_m = \frac{1}{\pi i} \int_{-\infty}^{\infty} \frac{(U(\xi))^{-1} - I_m}{\xi - \rho} d\xi, \quad \rho \in \mathbb{R} \setminus \{0\}.$$

Using (5.30), we get

$$U^*(\rho) - I_m = \frac{1}{\pi i} \int_{-\infty}^{\infty} \frac{U^*(\xi) - I_m}{\xi - \rho} d\xi, \quad \rho \in \mathbb{R} \setminus \{0\}.$$

On the other hand, it follows from (5.32), that

$$U^*(\rho) - I_m = -\frac{1}{\pi i} \int_{-\infty}^{\infty} \frac{U^*(\xi) - I_m}{\xi - \rho} d\xi, \quad \rho \in \mathbb{R} \setminus \{0\}.$$

Consequently, $U^*(\rho) - I_m = 0_m$ for real $\rho \neq 0$. Together with (5.31) this implies $U(\rho) \equiv I_m$, $\text{Im } \rho \geq 0$, $\rho \neq 0$. Thus, the solution of RP is unique. \square

Now consider the matrix Korteweg-de Vries equation

$$Q_t = 3QQ_x + 3Q_xQ - Q_{xxx}.$$

It is well known [25], that the evolution of the scattering data for it is described by the following relations:

$$S_+(\rho, t) = S_+(\rho, 0) \exp(8i\rho^3 t), \quad \rho_k(t) = \rho_k(0), \quad N_k^+(t) = N_k^+(0) \exp(8i\rho_k^3 t), \quad k = \overline{1, N}.$$

If the matrix function $D(\rho, 0)$ satisfies Condition B for the initial scattering data $J_+(0) = \{S_+(\rho, 0), \rho_k(0), N_k^+(0)\}$, then it solves RP for $G(\rho) = I_m - S_+^*(\rho, t)S_+(\rho, t)$, $\rho_k(t)$, $N_k^+(t)$ for every t . Since this solution is unique, it remains to check the estimates (4.1) for $R_{\pm}(x, t)$. The other requirements of Conditions A₊ and B are trivial.

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